COMMUNICATION

A NOTE ON COLOURED QUADRANGULATIONS

Kenneth A. BERMAN*

Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ont. N2L 3G1, Canada

Communicated by W.T. Tutte
Final form received 25 August 1979

Consider a quadrilateral drawn in the plane labelled ABCD counter-clockwise. Partition this quadrilateral into smaller quadrilaterals by selecting an even number of points (or no points) on each of the sides and any number of points in the interior. The resultant map is a planar *quadrangulation*, that is a planar map such that each face has valency 4 except possibly the outside face. (See [1]). Label the points on the side AB alternately A and B. Similarly label the points on the other three sides alternately B and C, C and D, D and A. Now, label the interior points in any way such that no two adjacent points (perimeter or interior) have the same label. Call the resultant coloured quadrangulation Q. There are 24 possible labellings of the small quadrilaterals, six of which use all four labels. In Fig. 1, an example of a coloured quadrangulation Q is given. Note that there is a small quadrilateral labelled ABCD counterclockwise.

The purpose of this note is to prove the following theorem.

Theorem. *The quadrangulation Q contains at least one small quadrilateral labelled* ABCD *counterclockwise.*

We first prove three lemmas.

Lemma 1. *For q a small quadrilateral of Q let $\delta_{AB}(q)$ denote the number of sides of q labelled* AB *minus the number of sides labelled* BA *when q is transversed in a counterclockwise direction. Then*

$$\sum_q \delta_{AB}(q) = 1 \tag{1}$$

where the summation is taken over all the small quadrilaterals of Q.

Let e be an edge of Q with ends labelled A and B. Consider the contribution of e to the summation on the left hand side of (1). If e is on the perimeter, then it

* Present address: Department of Pure Mathematics and Statistics, University of Cambridge, 16 Mill Lane, Cambridge CB2 1SB, England.

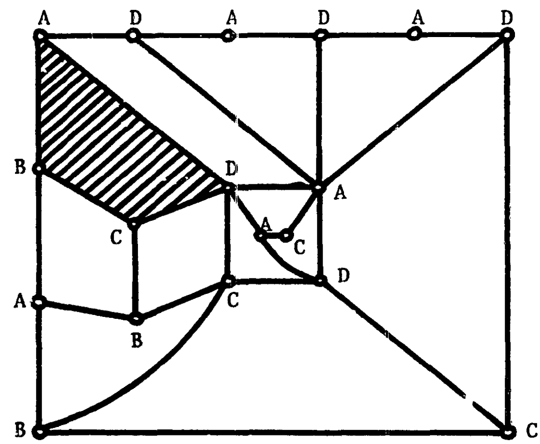
Q = 

Fig. 1.

belongs to exactly one small quadrilateral. Thus edge e is counted once positively if it is labelled AB when the perimeter polygon is transversed counterclockwise and once negatively if it is labelled BA. If e is an interior edge, then it belongs to exactly two quadrilaterals and is counted positively by one and negatively by the other. Therefore an edge contributes to the sum if and only if it lies on the perimeter. Lemma 1 follows since the number of perimeter edges labelled AB is one greater than the number labelled BA when the perimeter polygen is transversed counterclockwise.

Lemma 2. *For any small quadrilateral q of Q*

$$\delta_{\mathrm{AB}}(q)=\begin{cases} 1 & \textit{if } q \textit{ is labelled } \mathrm{ABCD} \textit{ or } \mathrm{ABDC} \textit{ counterclockwise,}\\ -1 & \textit{if } q \textit{ is labelled } \mathrm{BACD} \textit{ or } \mathrm{BADC} \textit{ counterclockwise,}\\ 0 & \textit{otherwise.}\end{cases} \tag{2}$$

Lemma 2 is easily verified.

Lemma 3. *Let $\{W, X, Y, Z\}=\{\mathrm{A},\mathrm{B},\mathrm{C},\mathrm{D}\}$ and let $n(WXYZ)$ be the number of small quadrilaterals of Q labelled $WXYZ$ counterclockwise. Then*

$$n(\mathrm{ABCD})-n(\mathrm{ADCB})=1. \tag{3}$$

It follows from Lemma 1 and Lemma 2 that

$$n(\mathrm{ABCD})+n(\mathrm{ABDC})-n(\mathrm{BACD})-n(\mathrm{BADC})=1. \tag{4}$$

By considering the edges labelled CD instead of the edges labelled AB we have the following analogous result

$$n(\mathrm{ABCD})+n(\mathrm{BACD})-n(\mathrm{ABDC})-n(\mathrm{BADC})=1. \tag{5}$$

Summing equations (4) and (5) and dividing by 2 yields

$$n(\mathrm{ABCD}) - n(\mathrm{BADC}) = 1$$

as asserted in the Lemma.

The Theorem follows immediately from identity (3) of Lemma 3.

References

[1] R.C. Mullin and P.J. Schellenberg, The enumeration of c-nets via quadrangulations, J. Combinatorial Theory 4 (1968) 259–276.

[2] E. Sperner, Einer Satz über Untermengen einer endlichen Menge, Math. Z. 27 (1928) 544–548.